

ON BROCKETT'S CONDITION FOR SMOOTH STABILIZABILITY AND ITS NECESSITY IN A CONTEXT OF NONSMOOTH FEEDBACK*

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Abstract. The necessity of Brockett's condition for stabilizability of nonlinear systems by smooth feedback is shown, by an argument based on properties of a degree for set-valued maps, to persist when the class of controls is enlarged to include discontinuous feedback.

Key words. degree, discontinuous control, nonlinear systems, nonsmooth feedback, set-valued maps, stabilizability

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1. Introduction. Consider the control system

$$(1) \quad \dot{x}(t) = f(x(t), u(t)), \quad x(0) = x^0 \in \mathbb{R}^N, \quad f(0, 0) = 0,$$

with $f : \mathbb{R}^N \times \mathbb{R}^M \rightarrow \mathbb{R}^N$ continuous. In the case of linear f , it is well known that the system is (globally) asymptotically null controllable if and only if it is stabilizable by (linear) feedback. Brockett [2] has shown that an analogous equivalence of (local) asymptotic null controllability and (nonlinear) feedback stabilizability does not hold for smooth (by which we mean C^1) nonlinear systems. In particular, he proved a result that implies the following necessary condition for (local) smooth stabilizability—henceforth referred to as Brockett's condition.

BROCKETT'S CONDITION. *Let $f \in C^1$. If (1) is C^1 stabilizable (in the sense that there exists a time-invariant C^1 feedback that renders $\{0\}$ both Lyapunov stable and an attractor), then the image of f contains an open neighbourhood of 0.*

If f is linear, that is, if $f(x, u) = Ax + Bu$, then the necessary condition for stabilizability is simply the requirement that $[A : B]$ be of full rank, and this is implied by asymptotic null controllability of the linear system. However, for general $f \in C^1$, (local asymptotic) null controllability of (1) does not imply that f has the above property, a (now classic) illustration is the case

$$f : \mathbb{R}^3 \times \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad (x, u) = (x_1, x_2, x_3, u_1, u_2) \mapsto (u_1, u_2, x_2u_1 - x_1u_2)$$

that defines a completely controllable bilinear system (1) for which $(0, 0, \epsilon) \notin \text{im}(f)$ for all $\epsilon \neq 0$, and so this system is not C^1 stabilizable.

Such examples are counterintuitive. It is tempting to conjecture that the “gap” between controllability and feedback stabilizability is due to the restriction to the class of smooth (C^1) time-invariant feedbacks. As in Sontag [11], the investigation readily extends to time-invariant feedbacks that are only locally Lipschitz (in fact, even this requirement is too strong, its consequence, *uniqueness of the solution* of the feedback-controlled initial-value problem, suffices as in [13]) and the gap is found to persist. Furthermore, Zabczyk [13] has shown that the necessity of Brockett's condition on f also persists when “stabilizability by time-invariant continuous feedback is interpreted in either of the following senses: (i) that of rendering $\{0\}$ a global attractor (which, of course, does not imply Lyapunov stability of $\{0\}$), or (ii) in the case of $n \leq 2$, that of rendering $\{0\}$ Lyapunov stable. Two possible avenues for further investigation suggest themselves naturally: (a) time-varying feedback and (b) discontinuous feedback. The former avenue has been followed by Coron [5]. In the case of $f(x, u) = \sum_{i=1}^M u_i f_i(x)$ with

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$f_i \in C^\infty(\mathbb{R}^N)$, he has established that the accessibility rank condition, $\dim \text{Lie}(\Phi)(x) = N$ for all $x \in \mathbb{R}^N \setminus \{0\}$ (where $\text{Lie}(\Phi)$ denotes the Lie algebra of vector fields generated by $\Phi = (f_1, \dots, f_M)$), is sufficient for the existence of T -periodic C^∞ feedbacks that globally asymptotically stabilize (1). In particular, this result applies to the example cited above. In the present paper, we take the second avenue and restrict to time-invariant feedbacks.

Discontinuous feedbacks arise naturally in many areas of control theory (see [7]) and practice (indeed, bang-bang or relay-type control actions permeate much of the early development of the field). It is not difficult to construct examples that fail to be locally asymptotically stabilizable by continuous feedback, but that are so stabilizable by discontinuous feedback. One such example is system (1) with $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $(x, u) \mapsto x + |x|u$. Therefore, the additional dynamic behaviours engendered by discontinuous feedbacks (that subsume the continuous case) raise the question of whether or not their adoption might close the controllability-stabilizability gap. Here this question is answered negatively. We show that with f only required to be continuous and to have property (2), below, and with the class of time-invariant feedbacks taken to be that of upper semicontinuous set-valued maps with nonempty convex and compact values (a class into which a wide variety of discontinuous strategies may be embedded and within which continuous feedbacks may be identified with the subclass of singleton-valued maps), the necessity of Brockett’s condition on f again persists.

2. Class of systems and statement of main result. We study systems of form (1) and assume only that continuous f has the property (see also Remark 1, below)

$$(2) \quad K \subset \mathbb{R}^M \text{ convex} \implies f(x, K) \subset \mathbb{R}^N \text{ convex.}$$

Evidently, (2) holds for systems that are linear in the control.

As *admissible feedback controls* for (1), we take the class \mathcal{K} of upper semicontinuous maps $x \mapsto k(x) \subset \mathbb{R}^M$ on \mathbb{R}^N , with nonempty convex and compact values and with $0 \in k(0)$. For example, in the case $M = 1$, discontinuous feedbacks of the form $x \mapsto \gamma(x)\text{sgn}(\xi(x))$, with γ and ξ continuous and such that the product $\gamma(0)\xi(0)$ is zero, fall within our framework if the signum function is interpreted as the upper semicontinuous set-valued map

$$v \mapsto \text{sgn}(v) := \begin{cases} \{+1\}, & v > 0, \\ [-1, 1], & v = 0, \\ \{-1\}, & v < 0. \end{cases}$$

For every feedback $k \in \mathcal{K}$, the map $x \mapsto f(x, k(x))$ is also upper semicontinuous with nonempty convex and compact values. Therefore, for each $x^0 \in \mathbb{R}^N$, the initial-value problem

$$(3) \quad \dot{x}(t) \in f(x(t), k(x(t))), \quad x(0) = x^0$$

has at least one solution (see [1, Thm. 2.1.3]), that is, a function $x : [0, \omega) \rightarrow \mathbb{R}^N$, with $x(0) = x^0$, that is absolutely continuous on compact subintervals and that satisfies the differential inclusion in (3) almost everywhere. Moreover, every solution x can be maximally extended. Furthermore, if x is bounded on its maximal interval of existence $[0, \omega)$, then $\omega = \infty$ (see, for example, [10]). We say that $\{z\}$ is an equilibrium of (3) if $0 \in f(z, k(z))$. Note that, for each $k \in \mathcal{K}$, $\{0\}$ is an equilibrium of (3).

In contrast with the smooth case, the property of uniqueness of the solution for the initial-value problem (3) clearly does not hold in our general nonsmooth framework. Implicit in the following definition is a notion of local asymptotic stability wherein we impose “equi-attractivity” of the equilibrium $\{0\}$. In essence, attraction to this equilibrium is required to be uniform with respect to nonunique solutions.

DEFINITION 1. A feedback control $k \in \mathcal{K}$ is said to be equi-asymptotically stabilizing for (1) if it renders the equilibrium $\{0\}$ of (3) equi-asymptotically stable in the sense that the following two properties hold.

(i) Lyapunov stability of the equilibrium $\{0\}$: for each $\rho > 0$, there exists $\delta > 0$ such that

$$\|x^0\| \leq \delta \implies \|x(t)\| < \rho \quad \text{for all } t \geq 0$$

for every maximal solution x of the initial-value problem (3).

(ii) Equi-attractivity of the equilibrium $\{0\}$: there exists $\delta > 0$ and, to each $\tau > 0$, there corresponds $T > 0$ such that

$$\|x^0\| \leq \delta \implies \|x(t)\| < \tau \quad \text{for all } t \geq T$$

for every maximal solution x of the initial-value problem (3).

Although the above definition is intrinsic to the problem, the following weaker (but somewhat artificial) property of the feedback is all that is required in the analysis.

DEFINITION 2. A feedback control $k \in \mathcal{K}$ is said to be equi-constricting for (1) if (3) has the following property. There exist scalars $\rho > \delta > \tau > 0$ and $T > 0$ such that

$$\|x^0\| \leq \delta \implies \|x(t)\| < \rho \quad \text{for all } t \geq 0 \quad \text{and} \quad \|x(t)\| < \tau \quad \text{for all } t \in [T, 2T]$$

for every maximal solution x of (3).

It is clear that, if $k \in \mathcal{K}$ is an equi-asymptotically stabilizing feedback for (1), then k is an equi-constricting feedback for (1). While the former concept is manifestly more natural from an applications viewpoint, the latter is considerably weaker. In particular, Definition 2 simply invokes the existence of *some* quadruple (ρ, δ, τ, T) , assuring the requisite properties. In essence, solutions of (3) are required only to be bounded uniformly with respect to initial data in some closed ball (of radius δ) and, on an interval $[T, 2T]$, to take their values in some smaller ball (of radius $\tau < \delta$).

The main result we will prove is the following.

THEOREM 1. Let f be continuous with property (2). If there exists an equi-constricting feedback control $k \in \mathcal{K}$ for (1), then the image of f contains an open neighbourhood of 0.

A simple modification to the proof of Theorem 1 will yield the following generalization of Brockett's condition.

COROLLARY 1. Let f be continuous with property (2). If there exists an equi-asymptotically stabilizing feedback control $k \in \mathcal{K}$ for (1), then, for each open neighbourhood \mathcal{N} of $0 \in \mathbb{R}^N$, $f(\mathcal{N} \times \mathbb{R}^M)$ contains an open neighbourhood of 0.

Remark 1. If \mathcal{K} is replaced by the class of C^1 feedbacks and attention is restricted to functions $f \in C^1$, then condition (2), which plays its role only in assuring that the right hand side of (3) takes convex values, may be removed; furthermore, the qualifier "equi" in Definition 2 is redundant. In this manner, Brockett's original result for smooth systems may be recovered as a special case of the above. It is in this sense that we regard Corollary 1 as a generalization of Brockett's condition.

The proof of Theorem 1, which is given in §4, is degree-theoretic in nature and similar in concept to the approaches of [8, §52], [11, §4.8], and [13, §2]. However, in the present nonsmooth setting, we first require some appropriate notion of degree for set-valued maps. This has been investigated by Cellina and Lasota [3] (see also [9], [12], [6]), and a distillation of results pertinent to our application is given in the next section.

3. Degree for set-valued maps. Here the objective is to reiterate, within the framework of [3], [12] but tailored to our immediate purpose, some results pertaining to degree for set-valued maps. The approach to defining degree for a (suitably regular) set-valued map F is via the Brouwer degree for single-valued approximate selections for F . With this in mind, some basic definitions and properties of upper semicontinuous maps and approximate selections (for details, see [1], [6]) are initially assembled.

3.1. Upper semicontinuous maps and approximate selections. For notational convenience, write $X := \mathbb{R}^N$. The ball of radius $r > 0$, centred at $c \in X$, will be denoted $B_r(c)$; when $c = 0$, we simply write B_r . For nonempty subsets U, V of a Banach space Y , define

$$d(y, V) := \inf_{v \in V} \|y - v\| \quad \text{for all } y \in Y, \quad \text{and} \quad d^*(U, V) := \sup_{u \in U} d(u, V).$$

Let $x \mapsto F(x) \subset X$, with domain $\text{dom}(F) = D \subset X$, have nonempty values. F is upper semicontinuous if it is upper semicontinuous at each $x \in D$: for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$F(w) \subset F(x) + B_\epsilon \quad \text{for all } w \in D \cap B_\delta(x).$$

If $C \subset D$ is compact and F is upper semicontinuous with compact values, then $F(C)$ is compact.

THEOREM 2 (Approximate selection theorem). *Let F be an upper semicontinuous map with domain $D \subset X$ and taking nonempty convex compact values in X . For each $\epsilon > 0$, there exists a locally Lipschitz single-valued function $f_\epsilon : D \rightarrow \text{co}(F(D))$ such that*

$$d^*(\text{graph}(f_\epsilon), \text{graph}(F)) < \epsilon.$$

(Any such f_ϵ will be referred to as an approximate selection for F .)

3.2. Construction and properties of degree. Initially, we recall *Brouwer degree* in the context of single-valued maps. As before, let $X := \mathbb{R}^N$. Henceforth, $\Omega \subset X$ is a bounded open set, with closure $\bar{\Omega}$ and boundary $\partial\Omega$. Let

$$\mathcal{M} = \{(f, \Omega, p) \mid X \supset \Omega \text{ open bounded, } f : \bar{\Omega} \rightarrow X \text{ continuous, } p \in X \setminus f(\partial\Omega)\},$$

then the Brouwer degree deg_B is the unique map $\mathcal{M} \rightarrow \mathbb{Z}$ with the following properties:

B-1. $\text{deg}_B(I, \Omega, p) = 1$ for all $p \in \Omega$;

B-2. If $\text{deg}_B(f, \Omega, p) \neq 0$, then $p = f(x)$ for some $x \in \Omega$;

B-3. (Homotopic invariance). If $h : [0, 1] \times \bar{\Omega} \rightarrow X$ and $q : [0, 1] \rightarrow X$ are continuous with $q(t) \notin h(t, \cdot)(\partial\Omega)$ for all $t \in [0, 1]$, then $\text{deg}_B(h(t, \cdot), \Omega, q(t))$ is independent of $t \in [0, 1]$;

B-4. (Odd mappings). If Ω contains, and is symmetric about, the origin in X and $f(-x) = -f(x)$ for all $x \in \partial\Omega$, then $\text{deg}_B(f, \Omega, 0)$ is odd (and so is nonzero).

The class of set-valued maps F , to which the ensuing construction [3], [12] of degree applies, are precisely those satisfying the hypotheses of Theorem 2: upper semicontinuous maps $x \mapsto F(x) \subset X$ from $\text{dom}(F) \subset X$ to the nonempty convex compact subsets of X .

For every open bounded Ω , with closure $\bar{\Omega} \subset \text{dom}(F) \subset X$, and every $p \in X \setminus F(\partial\Omega)$, we define an integer $\text{deg}(F, \Omega, p)$, the degree of F (with respect to the set Ω and point p).

3.2.1. Construction. Let $p \in X \setminus F(\partial\Omega)$ and let $F^{(p)}$ denote the map defined on compact $\bar{\Omega}$ by $x \mapsto F(x) - \{p\} = \{v - p \mid v \in F(x)\}$. By Theorem 2, for each $\epsilon > 0$ there exists an approximate selection f_ϵ for $F^{(p)}$. We first show that, for all $\epsilon > 0$ sufficiently small, every such approximate selection f_ϵ has no zeros in $\partial\Omega$. Suppose otherwise. Then there

exist sequences (ϵ_n) , (f_{ϵ_n}) , and $(x_n) \subset \partial\Omega$, with $\epsilon_n \downarrow 0$, $0 = f_{\epsilon_n}(x_n) \in \text{co}(F^{(p)}(\bar{\Omega}))$, and $0 \in F^{(p)}(y_n) + B_{\epsilon_n}$ for some $y_n \in \bar{\Omega}$ with $\|x_n - y_n\| < \epsilon_n$. By compactness of $\bar{\Omega}$, (y_n) has a convergent subsequence (that we do not relabel), with limit z say, and so $x_n \rightarrow z \in \partial\Omega$ as $n \rightarrow \infty$. By upper semicontinuity of $F^{(p)}$, for each $\epsilon > 0$, $0 \in F^{(p)}(z) + B_{\epsilon_n + \epsilon}$ for all n sufficiently large. Therefore, $0 \in \overline{F^{(p)}(z)} = F^{(p)}(z)$ and so $p \in F(z)$ with $z \in \partial\Omega$, a contradiction. It follows that for all $\epsilon > 0$ sufficiently small, $\text{deg}_B(f_\epsilon, \Omega, 0)$ is well defined for every approximate selection f_ϵ for $F^{(p)}$.

Let f_ϵ and g_ϵ be any two such approximate selections. Define the continuous function

$$h_\epsilon : [0, 1] \times \bar{\Omega} \rightarrow X, \quad (t, x) \mapsto tf_\epsilon(x) + (1 - t)g_\epsilon(x).$$

For all $\epsilon > 0$ sufficiently small, $h_\epsilon(t, \cdot)$ has no zeros in $\partial\Omega$ for every $t \in [0, 1]$. This can be argued (in a similar manner to above) by contradiction. Suppose otherwise; then there exist $t \in [0, 1]$, a sequence (ϵ_n) with $\epsilon_n \downarrow 0$, and a sequence $(x_n) \subset \partial\Omega$ such that

$$0 = h_{\epsilon_n}(t, x_n) = tf_{\epsilon_n}(x_n) + (1 - t)g_{\epsilon_n}(x_n) \in tF^{(p)}(y_n) + (1 - t)F^{(p)}(z_n) + B_{\epsilon_n}$$

for some $y_n, z_n \in \bar{\Omega}$ with $\|x_n - y_n\|, \|x_n - z_n\| < \epsilon_n$. By compactness of $\bar{\Omega}$, without loss of generality we may assume that $z_n \rightarrow z$ and so $y_n \rightarrow z$ and $x_n \rightarrow z \in \partial\Omega$ as $n \rightarrow \infty$. By upper semicontinuity of $F^{(p)}$ and convexity of its values, for each $\epsilon > 0$, $0 \in F^{(p)}(z) + B_{\epsilon_n + \epsilon}$ for all n sufficiently large and so $0 \in F^{(p)}(z)$, contradicting the fact that $p \notin F(\partial\Omega)$. Therefore, for all $\epsilon > 0$, $0 \notin h_\epsilon(t, \cdot)(\partial\Omega)$ for all $t \in [0, 1]$. Thus, by property B-3, $\text{deg}_B(h_\epsilon(t, \cdot), \Omega, 0)$ is independent of $t \in [0, 1]$, and so we may conclude that, for all $\epsilon > 0$ sufficiently small,

$$\text{deg}_B(f_\epsilon, \Omega, 0) = \text{deg}_B(h(1, \cdot), \Omega, 0) = \text{deg}_B(h(0, \cdot), \Omega, 0) = \text{deg}_B(g_\epsilon, \Omega, 0).$$

Simply stated, for all $\epsilon > 0$ sufficiently small, $\text{deg}_B(f_\epsilon, \Omega, 0)$ is well defined for every approximate selection f_ϵ and is independent of the particular selection chosen.

In summary, the above construction ensures that the following concept of degree for the set-valued map F is well defined:

$$\text{deg}(F, \Omega, p) := \lim_{\epsilon \downarrow 0} \text{deg}_B(f_\epsilon, \Omega, 0).$$

3.2.2. Properties.

THEOREM 3. *Let $x \mapsto F(x) \subset X$ be upper semicontinuous on compact $\bar{\Omega} \subset X$ with nonempty, convex, and compact values.*

(i) *If $q : [0, 1] \rightarrow X \setminus F(\partial\Omega)$ is continuous, then $\text{deg}(F, \Omega, q(t))$ is independent of $t \in [0, 1]$.*

(ii) *If $p \in X \setminus F(\partial\Omega)$ is such that $\text{deg}(F, \Omega, p) \neq 0$, then $p \in F(x)$ for some $x \in \Omega$.*

Proof. By the above construction, all degrees in the assertions of the theorem are well defined.

Assertion (i) is an immediate consequence of the construction together with B-3.

(ii) Because $\text{deg}_B(F, \Omega, p) \neq 0$, there exists a sequence (ϵ_n) , with $\epsilon_n \downarrow 0$, and an associated sequence (f_{ϵ_n}) of approximate selections for $F^{(p)}$ such that $\text{deg}_B(f_{\epsilon_n}, \Omega, 0) \neq 0$ for all n sufficiently large. By B-2, for each n sufficiently large, there exists $x_n \in \Omega$ such that $0 = f_{\epsilon_n}(x_n)$. By compactness of $\bar{\Omega}$, without loss of generality we may assume that $x_n \rightarrow x \in \bar{\Omega}$. Moreover, because the functions f_{ϵ_n} are approximate selections, for each n there exists $y_n \in \bar{\Omega}$, with $\|x_n - y_n\| < \epsilon_n$, such that

$$0 = f_{\epsilon_n}(x_n) \in F^{(p)}(y_n) + B_{\epsilon_n}.$$

Arguing as before (using semicontinuity of F and compactness of its values), it follows that $0 \in F^{(p)}(x)$ and so $p \in F(x)$. This proves assertion (ii) of the theorem. \square

4. Proof of the main result. We now turn attention to the proof of Theorem 1. Again write $X := \mathbb{R}^N$. Assume $k \in \mathcal{K}$ is an equi-constricting feedback for (1). Then there exist $\rho > \delta > \tau > 0$ and $T > 0$ such that

$$\|x^0\| \leq \delta \implies \|x(t)\| < \rho \quad \text{for all } t \geq 0 \quad \text{and} \quad \|x(t)\| < \tau \quad \text{for all } t \in [T, 2T]$$

for every maximal solution $x(\cdot)$ of (3).

Define the set-valued map F on X as follows:

$$(4) \quad F : x \mapsto \begin{cases} f(x, k(x)), & \|x\| \leq \rho, \\ f(\rho\|x\|^{-1}x, k(\rho\|x\|^{-1}x)), & \|x\| > \rho. \end{cases}$$

It is evident that F is upper semicontinuous with nonempty convex and compact values, and so $F(\overline{B_\rho}) \equiv F(X)$ is compact. By the construction in §3.2.1, for every open bounded set $\Omega \subset X$ and every $p \in X \setminus F(\partial\Omega)$, $\text{deg}(F, \Omega, p)$ is well defined.

Consider the initial-value problem

$$(5) \quad \dot{x}(t) \in F(x(t)), \quad x(0) = x^0.$$

By compactness of $F(X)$ we may deduce that, for each $x^0 \in X$, every solution of (5) has maximal interval of existence $\mathbb{R}^+ := [0, \infty)$. Observe that, for each x^0 with $\|x^0\| \leq \delta$, the set of maximal solutions of (5) is precisely the set of maximal solutions of (3).

Write $\Omega^0 := B_\delta$, with closure $\overline{\Omega^0}$. By the equi-constricting property, the annulus $\overline{\Omega^0} \setminus B_\tau$ cannot contain an equilibrium of (5) (or, equivalently, a point x such that $0 \in F(x)$). Therefore, $0 \notin F(\partial\Omega^0)$ and $\text{deg}(F, \Omega^0, 0)$ is well defined. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of locally Lipschitz approximate selections for F with

$$d^*(\text{graph}(f_n), \text{graph}(F)) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and such that $\text{deg}(F, \Omega^0, 0) = \text{deg}_B(f_n, \Omega^0, 0)$ for all n .

Write $I := [0, 2T]$ and $Y := C(I; X)$ with the uniform norm. On $\overline{\Omega^0}$ we define the map

$$\mathcal{F} : x^0 \mapsto \{x \in Y \mid \dot{x}(t) \in F(x(t)) \text{ a.e., } x(0) = x^0\}.$$

For each n , define the map $\phi_n : \overline{\Omega^0} \rightarrow Y$ as follows: $\phi_n(x^0)$ is the unique element x of Y such that

$$\dot{x}(t) = f_n(x(t)) \quad \text{for all } t \in I, \quad \text{and} \quad x(0) = x^0.$$

By the classical theory of ordinary differential equations, the map $(t, x^0) \mapsto (\phi_n(x^0))(t)$ is continuous.

We claim that, for every $\epsilon > 0$,

$$d^*(\text{graph}(\phi_n), \text{graph}(\mathcal{F})) < \epsilon \quad \text{for some } n.$$

Suppose otherwise. Then there exist $\epsilon > 0$ and a sequence $(x_n^0) \subset \overline{\Omega^0}$ such that

$$d((x_n^0, \phi_n(x_n^0)), \text{graph}(\mathcal{F})) \geq \epsilon \quad \text{for all } n.$$

For notational convenience, we write $x_n = \phi_n(x_n^0)$. Arguing as in the first proof of Theorem 2.1.3 of [1] (see also [4, Thm 3.1.7]) and extracting a subsequence if necessary, we may assume that $(x_n) \subset Y$ converges uniformly to an absolutely continuous function $x : I \rightarrow X$, $x(0) =$

$x^0 \in \overline{\Omega^0}$, satisfying $\dot{x}(t) \in F(x(t))$ almost everywhere, whence the following contradiction: $(x_n^0, x_n) \rightarrow (x^0, x) \in \text{graph}(\mathcal{F})$ as $n \rightarrow \infty$.

Let $0 < \epsilon < \delta - \tau$ and let m be such that $d^*(\text{graph}(\phi_m), \text{graph}(\mathcal{F})) < \epsilon$. We assert that

$$(6) \quad \text{for all } x^0 \in \overline{\Omega^0}, \quad (\phi_m(x^0))(t) \in \Omega^0 \quad \text{for all } t \in [T, 2T].$$

This may be shown as follows. Let $x^0 \in \overline{\Omega^0}$ be arbitrary. There exists $y^0 \in \overline{\Omega^0}$, with $\|x^0 - y^0\| < \epsilon$, and $y \in \mathcal{F}(y^0)$ such that $\|(\phi_m(x^0))(t) - y(t)\| < \epsilon$ for all $t \in I$. Because the set $\{y(t) | y \in \mathcal{F}(\overline{\Omega^0})\}$ lies in the ball B_τ for all $t \in [T, 2T]$, the assertion must hold.

Define a function $h : [0, 1] \times \Omega^0 \rightarrow X$ by

$$h(s, x^0) := \begin{cases} f_m(x^0), & s = 0, \\ \frac{1}{s}[(\phi_m(x^0))(sT) - x^0], & 0 < s \leq 1. \end{cases}$$

That h is continuous is readily verified. Furthermore, $h(s, x^0) \neq 0$ for all $(s, x^0) \in [0, 1] \times \partial\Omega^0$ by the following argument. Suppose $h(0, x^0) = f_m(x^0) = 0$ for some $x^0 \in \partial\Omega^0$. Then $(\phi_m(x^0))(t) = x^0 \in \partial\Omega^0$ for all $t \in I$, which contradicts (6). Now suppose $h(s, x^0) = 0$ for some $(s, x^0) \in (0, 1] \times \partial\Omega^0$. Then $(\phi_m(x^0))(nsT) = x^0 \in \partial\Omega^0$ for all $n \in \mathbb{N}$ with $ns \leq 2$.

In particular, there exists $n \in \mathbb{N}$ such that

$$1 \leq ns \leq 2 \quad \text{and} \quad (\phi_m(x^0))(nsT) = x^0 \in \partial\Omega^0.$$

This contradicts (6).

We have now established h as a homotopic connection of the functions f_m and

$$g_m : x^0 \mapsto (\phi_m(x^0))(T) - x^0.$$

It is evident that $h_0 : [0, 1] \times \overline{\Omega^0}, (s, x^0) \mapsto (1 - s)g_m(x^0) - sx^0$ defines a homotopic connection of g_m and the odd map $x^0 \mapsto -x^0$. By properties B-3 and B-4, we may conclude that

$$\text{deg}(F, \Omega^0, 0) = \text{deg}_B(f_m, \Omega^0, 0) = \text{deg}_B(g_m, \Omega^0, 0) \neq 0.$$

Since $0 \notin F(\partial\Omega^0)$, $d(0, F(x)) > 0$ for all $x \in \partial\Omega^0$. Next, we show that $x \mapsto d(0, F(x))$ is lower semicontinuous on $\partial\Omega^0$. Let $x \in \partial\Omega^0$ be arbitrary and let $(x_n) \subset \partial\Omega^0$ be a convergent sequence with limit x . Let subsequence (x_{n_k}) be such that

$$\lim_{k \rightarrow \infty} d(0, F(x_{n_k})) = \liminf_{n \rightarrow \infty} d(0, F(x_n)).$$

For each k , let y_k be a minimizer of $\|\cdot\|$ over compact $F(x_{n_k})$, that is, $\|y_k\| = d(0, F(x_{n_k}))$. By upper semicontinuity of F , for each $\epsilon > 0$ we have $y_k \in F(x_{n_k}) \subset F(x) + B_\epsilon$ for all k sufficiently large. By compactness of $F(x)$, it follows that (y_k) has a convergent subsequence (that we do not relabel) with limit $y \in F(x)$, whence

$$d(0, F(x)) = \min_{v \in F(x)} \|v\| \leq \|y\| = \lim_{k \rightarrow \infty} \|y_k\| = \liminf_{n \rightarrow \infty} d(0, F(x_n)).$$

Thus, $x \mapsto d(0, F(x))$ is positive-valued and lower semicontinuous on compact $\partial\Omega^0$ and so attains a positive minimum value thereon. We may now conclude the existence of a scalar $\mu > 0$ such that $p \notin F(\partial\Omega^0)$ for all $p \in B_\mu$. By Theorem 3(i) we deduce that, for every such p ,

$$\text{deg}(F, \Omega^0, p) = \text{deg}(F, \Omega^0, 0) \neq 0.$$

Therefore, by Theorem 3(ii), for each $p \in B_\mu$ there exists $x \in \Omega^0 = B_\delta$ such that $p \in F(x) = f(x, k(x))$. It immediately follows that each $p \in B_\mu$ is the image, under f , of some point $(x, u) \in B_\delta \times \mathbb{R}^M$. This completes the proof of Theorem 1.

It remains only to prove Corollary 1. Let \mathcal{N} be any open neighbourhood of $0 \in X$ and let $\rho > 0$ be such that $B_\rho \subset \mathcal{N}$. Let $k \in \mathcal{K}$ be equi-asymptotically stabilizing. Then there exist scalars $T > 0$ and δ, τ , with $0 < \tau < \delta < \rho$, such that the equi-constricting property of Definition 2 holds. Now, arguing exactly as in the proof of Theorem 1, it follows that $f(B_\delta \times \mathbb{R}^M)$ (and so, a fortiori, $f(\mathcal{N} \times \mathbb{R}^M)$) contains an open neighbourhood of $0 \in X$.

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